

Fermi coordinates and modified Franklin transformation : A comparative study on rotational phenomena

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Abstract

Applying a relativistic rotational transformation to study and analyze rotational phenomena, instead of the rotational transformations based on consecutive Lorentz transformations and Fermi coordinates, leads to different predictions. In this article after a comparative study between Fermi metric of a uniformly rotating observer and the spacetime metric in a rotating frame obtained through the modified Franklin transformation, we consider rotational phenomena including transverse Doppler effect and Sagnac effect in both formalisms and compare their predictions. We also discuss length measurements in the two formalisms.

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I. INTRODUCTION

Rotation and rotating phenomena have always puzzled people and looking at the history of the theory of relativity (both special and general) it seems that rotating observers had a key role in forming Einstein's thoughts on non-inertial observers and their role in shaping the principles and formulation of the relativity theory [1]. A general feature in studying rotational phenomena is the coordinate transformation between inertial (non-rotating) and rotating observers. This is not only a matter of convenience in using the right coordinates but also a matter of getting the most plausible interpretation for the measurements made by different observers and their relations. It is expected that the relation between the spacetime measurements in the two frames (rotating and non-rotating) to be different if one employs different coordinate transformations between them. Different coordinate transformations are also expected to result in spatially different flat spacetime metrics in the rotating frame. Kinematically, the common practice in treating rotational phenomena is the employment of the so called Galilean rotational transformation (GRT) between the rotating and non-rotating frames. But this is questioned by many authors on the grounds that for relativistic rotational velocities one needs a relativistic rotational transformation (RRT), very much in the same way that Lorentz transformation replaces the Galilean transformation among inertial frames at relativistic velocities. There have been a few proposals for RRT dating back to the 1920s and the introduction of the first relativistic rotational transformation by Philip Franklin [2]. On the other hand the usual approach to the problem of physical phenomena in accelerated frames (of which the uniformly rotating frame is a special case), in flat and curved spacetimes, employs the so called hypothesis of locality [3]. This hypothesis asserts that accelerated observers are instantaneously equivalent to hypothetical inertial observers having the velocity of the accelerated observer at that moment. Therefore to find the coordinate transformation between two different positions of the accelerated observer on its worldline, one uses Lorentz transformations between the corresponding hypothetical inertial frames and a reference inertial frame. For example consider an observer at a given radius on a uniformly rotating platform. The coordinates that this observer assigns to an event at its two different rotational positions A and B around the origin O is found by a Lorentz transformation from the hypothetical inertial frame at A to the central inertial frame O , followed by a Lorentz transformation from O to the hypothetical inertial frame at B . This

is the same process which leads to the so called Thomas precession.

Therefore to study physical phenomena in the special case of a uniformly rotating frame one can either use a RRT or employ the hypothesis of locality. To emphasize once more the fundamental difference between the two approaches, we note that at the heart of the two approaches lies two different kinematical transformations, in the former it is a relativistic rotational transformation (RRT) such as the modified Franklin transformation whereas in the latter, the hypothesis of locality would allow one to use consecutive Lorentz transformations (LT) among hypothetical inertial observers which are instantaneously equivalent to the accelerated one at each moment on its world line.

It should be noted that it is the implicit application of the same hypothesis which leads to the so called Fermi coordinates and Fermi metric that a general accelerated spinning observer, carrying a tetrad, would assign to his/her reference frame [4]. It is interesting to find out that Fermi coordinates were introduced in exactly the same year that Franklin transformation was introduced [5]. Our main goal here is to study these two approaches and compare their predictions for the well known rotational phenomena. The outline of the paper is as follows. In the next section we introduce the Fermi coordinates of an accelerated spinning observer and in section III the same coordinates are used to find the Fermi metric in a rotating frame. In section IV modified Franklin transformation for eccentric uniformly rotating observers are introduced and the spacetime metric, based on this transformation, is given in a rotating frame. In section V transverse Doppler effect and Sagnac effect are studied comparatively by applying these two different approaches. In the same section the relation between length measurements in rotating and inertial frames will be discussed in both approaches. In what follows Roman indices run from 1 to 3 while Greek ones run from 0 to 3 and our metric signature is $(-, +, +, +)$.

II. ACCELERATED, ROTATING OBSERVERS AND FERMI COORDINATES

As pointed out in [4] “*it is very easy to put together the words “the coordinate system of an accelerated observer” but it is much harder to find a concept these words might refer to*” and it gets even harder, at least conceptually, if one wishes to extend it to *accelerated observers* in curved spacetimes. One could assign a coordinate system to an accelerated spinning observer, who carries an orthonormal tetrad, both in flat and curved spacetimes.

Although the formalism, to first order in the spatial coordinates, looks the same in flat and curved spacetimes, it should be noted that the main difference is in the size of the region in which the coordinate is applicable. In the case of flat spacetime it could be applied to a region within a finite distance from the observer but in the case of curved spacetime it is restricted to an infinitesimal neighbourhood of the tetrad's origin on the observer's world line [21]. In this approach setting the origin of the accelerated observer's frame (S) on the observer's world line, the orthonormal tetrad $e^\mu(\tau)$ (consisting of one timelike and three spacelike vectors) carried by the observer is Fermi-walker transported and the coordinate transformation between an inertial (Laboratory) observer and the accelerated observer is given by [4],

$$x'^\mu = x^k [e_k(\tau)]^\mu + Z^\mu(\tau) \quad k = 1, 2, 3 \quad (1)$$

in which τ is the observer's proper time and $Z^\mu(\tau)$ is its worldline relative to an inertial frame. x'^μ and x^k are the coordinates assigned to an event in the inertial frame (S') and in the accelerated observer's local frame (S) respectively. Now if the observer has a spatial rotation (i.e carrying a spinning tetrad), its tetrad is not Fermi-Walker transported but is transported according to following rule

$$\frac{d[e_\alpha]^\mu}{d\tau} = -\Omega^{\mu\nu} [e_\alpha]_\nu \quad (2)$$

where α is the tetrad (Lorentz) index and

$$\Omega^{\mu\nu} = a^\mu u^\nu - a^\nu u^\mu + u_\alpha \Omega_\beta{}^\epsilon \epsilon^{\alpha\beta\mu\nu} \quad (3)$$

is composed of two parts, the first part made out of u^μ and a^μ (i.e the 4-velocity and 4-acceleration of the observer), indicates the Fermi-Walker part while the second part (last term in (3)), denoting the spatial rotation, includes the observer's 4-rotation Ω^μ . The first order expression for the metric near the observer's world line is given by the so called Fermi metric [5],

$$ds^2 = -(1 + 2a''_l x^l) dx^{0^2} - 2(\epsilon_{jkl} x^k \Omega''^l) dx^0 dx^j + \delta_{ij} dx^i dx^j \quad (4)$$

in which \mathbf{a}'' and $\mathbf{\Omega}''$ are the observer's acceleration and (spin) rotation measured in a comoving inertial frame (S'') whose velocity is momentarily the same as that of the accelerating

observer. In [7] by extending this method to the second order in x^i , the metric in an accelerated spinning frame in curved spacetime, is derived as follows,

$$ds^2 = -dx^{02}[1 + 2a''_j x^j + (a''_l x^l)^2 + (\Omega''_l x^l)^2 - \Omega''^2 x_l x^l + R_{0l0m} x^l x^m] \\ + 2dx^0 dx^i (\epsilon_{ijk} \Omega''^j x^k - \frac{2}{3} R_{0lim} x^l x^m) + dx^i dx^j (\delta_{ij} - \frac{1}{3} R_{iljm} x^l x^m) \quad (5)$$

which in flat spacetime reduces to

$$ds^2 = -dx^{02}[1 + 2a''_l x^l + (a''_l x^l)^2 + (\Omega''_l x^l)^2 - \Omega''^2 x_l x^l] \\ + 2dx^0 dx^i (\epsilon_{ijk} \Omega''^j x^k) + dx^i dx^j \delta_{ij}. \quad (6)$$

In [8], looking for a generalization of the Lorentz transformation to the case of accelerated rotating observers (with a time-dependent velocity), a nonlinear coordinate transformation was introduced which not only incorporates the Thomas precession but also leads to the metric (6) exactly. Two of the main properties of metric (6) are as follows [9, 10],

I- In the absence of any linear acceleration ($a = 0$) this metric (in the cartesian coordinates) reduces to

$$ds^2 = -[1 - (x^2 + y^2)\Omega^2]dt^2 + dx^2 + dy^2 + dz^2 - 2y\Omega dxdt + 2x\Omega dydt \quad (7)$$

which is the Galilean rotational metric assigned to the flat spacetime by an observer at $r = 0$ with constant angular velocity Ω . The same metric in cylindrical coordinates (t, r, z, ϕ) is given by [11]

$$ds^2 = -(1 - r^2\Omega^2)dt^2 + dr^2 + dz^2 + r^2 d\phi^2 + 2\Omega r^2 d\phi dt \quad (8)$$

II- In the absence of any spatial rotation ($\Omega = 0$), as expected, this metric reduces to that of Rindler metric

$$ds^2 = -dx^{02}[1 + 2a''_j x^j + (a''_l x^l)^2] + dx^i dx^j \delta_{ij} \quad (9)$$

which is the metric of the flat spacetime in the proper frame of a uniformly accelerated observer.

III. UNIFORMLY ROTATING OBSERVERS IN FERMI COORDINATES

Having discussed the general case of accelerated spinning observer and its spacetime metric in the previous section, here we are interested in the particular case of an observer who

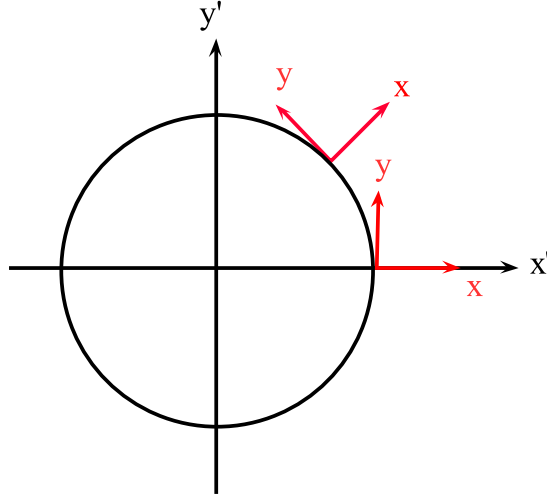


FIG. 1: Local coordintes of a rotating observer with axes x and y which coincide with the radius and tangent of the circular path respectively.

moves on a circular path such as the one on a uniformly rotating disk. In [9] (based on the hypotheis of locality) and in [10] (based on the formulation in [8]) coordinate transformation between such an observer and an inertial (laboratory) observer has been introduced. In their setup the origin of the rotating frame is on the rim of the circular path and its axes are oriented suh that the Y-axis is always tangent to the circular path. Also at $t = 0$ the axes of the inertial and accelerated frames are parallel (Figure 1). Therefore as in the case of Fermi metric for an accelerated, spinning observer, the origin of the rotating observer's frame is on the observer's world line and is attached rigidly to the rotating disk. It may seem that this special case reduces the generality of the problem, but it should be noted that such construction of frames are important as they are the ones which are related to the real experimental setups such as in the case of observers/detectors on the circumference of a uniformly rotating disk. In [9] after introducing the corresponding tetrad for such an observer and using the same method employed in obtaining the Fermi metric in [4], the following coordinate transformation between inertial (primed) and uniformly rotating (unprimed) frames, with angular velocity Ω [22]

$$\begin{aligned} ct &= \gamma^{-1}(ct' - \beta\gamma y) & x &= x' \sin(\gamma\Omega ct) + y' \cos(\gamma\Omega ct) - R \\ y &= \gamma^{-1}[x' \cos(\gamma\Omega ct) + y' \sin(\gamma\Omega ct)] & , \quad z &= z' \end{aligned} \quad (10)$$

in which $\gamma = (1 - \beta^2)^{-1/2}$, $\beta = \frac{R\Omega}{c}$ and R is the radius of the circular path. By the above transformation the circular path $x'^2 + y'^2 = R^2$ in the inertial frame, is an ellipse $(x + r)^2 + \gamma^2 y^2 = R^2$ with a contracted circumference (refer to section V) in the rotating observer's frame [12]. Using the general Lorentz transformation introduced in [8], the inverse of the above coordinate transformations are given in [10] as follows,

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & \gamma\Omega R/c^2 & 0 \\ 0 & \cos(\Omega t) & -\gamma \sin(\gamma\Omega t) & 0 \\ 0 & \sin(\gamma\Omega t) & \gamma \cos(\gamma\Omega t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} + R \begin{pmatrix} 0 \\ \cos(\gamma\Omega t) - 1 \\ \sin(\gamma\Omega t) \\ 0 \end{pmatrix} \quad (11)$$

which are, up to a constant shift in the x-direction, the same as coordinate transformations given in [12]. Differentiating the above coordinates and substituting them into the Minkowski metric of the inertial observer, the line element in the rotating frame is given by,

$$ds^2 = -\gamma^2[1 - (R + x)^2\Omega^2 - \Omega^2\gamma^2 y^2]dt^2 + dx^2 + dy^2 + dz^2 - 2y\Omega dxdt + 2x\Omega dydt \quad (12)$$

Comparison with the general Fermi metric (6) reveals the following 3-vectors [10]

$$a''^l = (-\gamma^2 R\Omega^2, 0, 0) \quad , \quad \Omega''^l = (0, 0, \gamma^2\Omega) \quad (13)$$

as the observer's acceleration and angular velocity measured by the comoving inertial frame. In other words for an observer with the above 3-acceleration and 3-angular velocity (6) concludes (12). Since the acceleration is proportional to R , it can be seen that at $R = 0$ the metric (12) reduces to that in the rotating frame (7) obtained through the application of the Galilean rotational transformation.

Limitations in length measurements by a uniformly rotating observer in this construction are discussed in [12]. Although existence of a rotating disk is not considered in the construction of the Fermi metric, but uniformly rotating observer as introduced here (with equal spinning and orbiting angular velocities) could be realized in an experimental setup with a detector at a non-zero radius on a rotating disk. As we will discuss later, this is basically the experimental setup used to investigate transverse Doppler effect as a rotational phenomena.

IV. UNIFORMLY ROTATING ECCENTRIC OBSERVERS AND MODIFIED FRANKLIN TRANSFORMATION

In [13], looking for a consistent relativistic rotational transformation between an inertial observer (frame S') and an observer at a non-zero radius (eccentric observer) on a uniformly rotating disk (frame S), the following modification of the so called Franklin transformation (in cylindrical coordinates) was introduced,

$$\begin{aligned} t &= \cosh(\Omega R/c)t' - \frac{R}{c} \sinh(\Omega R/c)\phi' \quad ; \quad r = r' \\ \phi &= \cosh(\Omega R/c)\phi' - \frac{c}{R} \sinh(\Omega R/c)t' \quad ; \quad z = z' \end{aligned} \quad (14)$$

in which Ω is the uniform angular velocity of the disk and R is the radial position of the observer on the disk. Note that the origin of the rotating frame S is chosen to be at the center of the rotating disk so that both inertial and rotating frames assign the same radial coordinate to the events (Fig. 2). The corresponding metric in the rotating observer's frame is given by

$$\begin{aligned} ds^2 &= -c^2 \cosh^2 \beta \left(1 - \frac{r^2}{R^2} \tanh^2 \beta\right) dt^2 + dr^2 + r^2 \cosh^2 \beta \\ &\left(1 - \frac{R^2}{r^2} \tanh^2 \beta\right) d\phi^2 - 2cR \sinh \beta \cosh \beta \left(1 - \frac{r^2}{R^2}\right) dt d\phi + dz^2 \end{aligned} \quad (15)$$

As in the case of Franklin transformation, this is the flat spacetime metric with non-Euclidean spatial sector. But, unlike Franklin transformation, it reduces to the spacetime metric obtained through Galilean rotational transformation in the limit $\beta \rightarrow 0$, i.e close to the rotation axis [13] where the rotational velocity is non-relativistic. Also as in the case of Fermi metric, at the position of the observer i.e, $r = R$, this metric reduces to that of spatially Euclidean Minkowski metric in cylindrical coordinates. To compare the above metric for a rotating observer with that obtained for the same observer in Fermi coordinates, we rewrite it in the cartesian coordinate as follows,

$$\begin{aligned} ds^2 &= -c^2 \left[\cosh^2 \beta - \frac{x^2 + y^2}{R^2} \sinh^2 \beta \right] dt^2 + \left[\frac{x^2}{r^2} + \left(\cosh^2 \beta - \frac{R^2}{r^2} \sinh^2 \beta \right) \frac{y^2}{r^2} \right] dx^2 \\ &+ \left[\frac{y^2}{r^2} + \left(\cosh^2 \beta - \frac{R^2}{r^2} \sinh^2 \beta \right) \frac{x^2}{r^2} \right] dy^2 - \sinh^2 \beta \left(1 - \frac{R^2}{r^2}\right) \frac{xy}{r^2} dx dy \\ &+ 2R \sinh \beta \cosh \beta \left(\frac{1}{r^2} - \frac{1}{R^2} \right) y dt dx - 2R \sinh \beta \cosh \beta \left(\frac{1}{r^2} - \frac{1}{R^2} \right) x dt dy + dz^2 \end{aligned} \quad (16)$$

Since both metrics at the position of the observer reduce to the spatially Euclidean flat space metric, to compare them, the above line element is expanded around the position of

the observer at $(x_0 = R, y_0 = 0)$ (Fig2), leading to the following non-zero components of the metric,

$$g_{00} = -1 + \frac{2 \sinh^2 \beta}{R} \xi_1 + \frac{\sinh^2 \beta}{R^2} (\xi_1^2 + \xi_2^2) \quad , \quad g_{01} = 0 \quad , \quad g_{02} = \frac{2 \sinh \beta \cosh \beta}{R} \xi_1$$

$$g_{11} = 1 \quad , \quad g_{22} = 1 + \frac{-2 \sinh^2 \beta}{R} \xi_1 \quad , \quad g_{33} = 1 \quad (17)$$

in which $\xi_i = (\xi_1, \xi_2)$ represents a small (Cartesian) displacement from the position of the rotating observer at O . By expanding $\sinh \beta$ and $\cosh \beta$ in (17) and also γ in (12) to the second order in $\beta = \frac{R\Omega}{c}$ it can easily be seen that the time-time components g_{00} in the two metrics are the same (identifying ξ_i with $x_i = (x, y)$ in (12)) but the other components differ. Consequently it is expected that the rotational effects originating from the time-time component of the metric in both approaches lead to the same predictions but those arising from other components of the two metrics, specially the cross terms, should lead to different predictions. These matters will be discussed in the next section where we comparatively study rotational phenomena in the two approaches introduced here and in the previous section.

V. APPLICATION TO ROTATIONAL PHENOMENA

In the last two sections we have introduced two different coordinate transformation between an inertial and a rotating frame. They were based on consecutive Lorentz transformations and an exact relativistic rotational transformation (modified Franklin transformation) respectively. Consequently they led to two different spatially non-Euclidean flat spacetime metrics in the rotating frame, with the same time-time component to the second order in $\beta = \frac{\Omega R}{c}$. Here we are going to study comparatively the application of these transformations to physical phenomena related to rotating systems. In each case, the results obtained by employing the two coordinate systems are expanded in terms of the parameter $\beta = \frac{\Omega R}{c}$ to see how the results differ from the classical (non-relativistic) case in which GRT or its equivalent metric is used. In other words the comparison is made between the results obtained in the two relativistic approaches at the classical limit $\Omega R \ll c$.

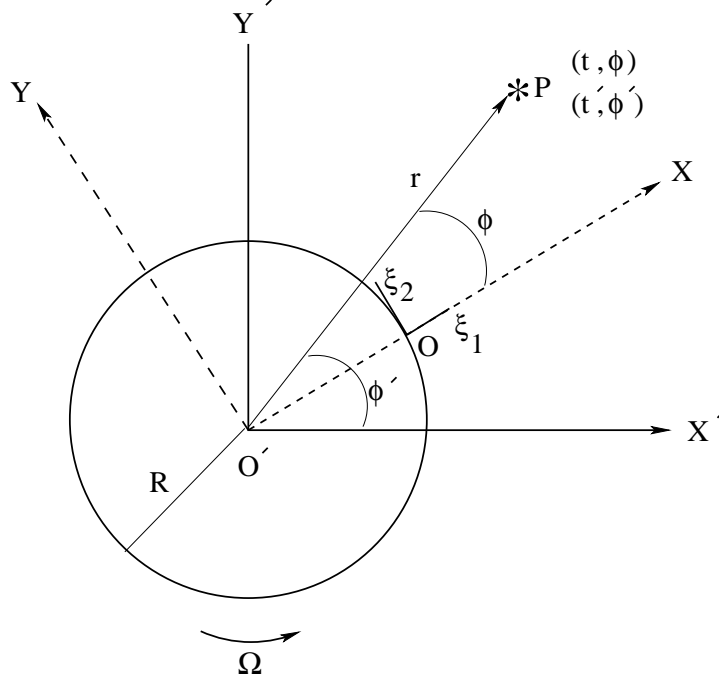


FIG. 2: Inertial frame (X', Y') and local frame (X, Y) of a rotating observer O , at the rim of a uniformly rotating disk with radius R . In cylindrical coordinates, an event P has temporal and angular coordinates (t, ϕ) and (t', ϕ') in rotating and inertial frames respectively.

A. Transverse Doppler Effect

Transverse Doppler effect is a direct consequence of the time dilation in special relativity. In the simplest setup, an observer moving on a rotating disk will measure the frequency of a light signal sent from a centrally located source. So to examine the effect of employment of a relativistic rotational transformation or consecutive Lorentz transformations based on the hypothesis of locality, one should examine the relation between the time intervals in the non-rotating and rotating coordinates. In this regard, the relation between the time intervals in an inertial observer's frame and a uniformly rotating one in the formalism based on the Fermi coordinates is given by the first equation in (10). Looking at that equation it is obvious that the transverse Doppler effect in this approach is the same as what one gets in special relativity [19]. Now if the source and receiver are both on the rotating disk at radii R_1 and R_2 respectively, the ratio of the emitted frequency to that received is given by

$$\frac{\nu_1}{\nu_2} = \frac{\gamma_1}{\gamma_2} \approx 1 + \frac{\Omega^2}{2c^2}(R_1^2 - R_2^2) + \frac{\Omega^4}{c^4}\left(\frac{3}{8}R_1^4 - \frac{1}{8}R_2^4 - \frac{1}{4}R_1^2R_2^2\right) \quad (18)$$

on the other hand using the same setup in the context of the MFT, by the first equation in (15), one arrives at the following result

$$\frac{\nu_1}{\nu_2} = \frac{\cosh \beta_1}{\cosh \beta_2} \approx 1 + \frac{\Omega^2}{2c^2}(R_1^2 - R_2^2) + \frac{\Omega^4}{c^4}\left(\frac{1}{24}R_1^4 - \frac{5}{24}R_2^4 - \frac{1}{4}R_1^2 R_2^2\right) \quad (19)$$

which differs from the previous result in the third term which is of fourth order in $\beta = \frac{\Omega R}{c}$. The same result could also be obtained by using the corresponding metric (15) in the rotating frame and noting that the proper times at $r_0 = 0$ and $r_1 = R$ are related by

$$d\tau_0^2 = \cosh^2 \beta_1 d\tau_1^2 \quad (20)$$

in which $\beta_1 = \frac{\Omega R_1}{c}$. So for two observers sitting at two different radii $r = R_1$ and $r_2 = R_2$, the frequencies measured in terms of their proper times are related by,

$$\frac{\nu_1}{\nu_2} \equiv \frac{d\tau_2}{d\tau_1} = \frac{\cosh \beta_1}{\cosh \beta_2} \quad (21)$$

This verifies our expectation on the rotational effects related to the time-time component of the metric in the rotating frame in the two formalisms. In either of the relations (18) and (19) one could set $R_1 = 0$ and $R_2 = R_0$ (with R_0 the radius of the rotating disk) to find out the frequency ratio for the case in which the source and receiver are at the center and rim of the disk respectively so that (18) and (19) reduce to

$$\frac{\nu_1}{\nu_2} = \sqrt{1 - \frac{R_0^2 \Omega^2}{c^2}} \approx 1 - \frac{\Omega^2}{2c^2} R_0^2 - \frac{1}{8} \frac{\Omega^4}{c^4} R_0^4 \quad (22)$$

and

$$\frac{\nu_1}{\nu_2} = \sqrt{1 - \tanh^2 \frac{R_0 \Omega}{c}} \approx 1 - \frac{\Omega^2}{2c^2} R_0^2 - \frac{5}{24} \frac{\Omega^4}{c^4} R_0^4 \quad (23)$$

respectively. In this way one could observe that in the formalism based on MFT, the frequency ratio arises from the same relation as in the special relativistic case but now with the non-linear velocity $v = c \tanh \beta$ replacing the Galilean velocity $v = R_0 \Omega$ [13]. This is also the original experimental set up in Kündig's experiment in which the mössbauer effect was used to verify transverse Doppler effect [14]. There has been some controversy over the interpretation of the null result on the existence of this effect at microwave frequency in a more recent experiment (refer to [15] and references therein for a further discussion in this regard).

B. Sagnac Effect

Perhaps one of the most famous rotational effects is the so called Sagnac effect [16], in which an interferometer on a rotating platform measures the effect of rotation on the phases of counter rotating photon beams. For two such beams started at the same point on a rotating platform with uniform angular velocity Ω , the difference in their arrival time to the initial point, as measured by an inertial frame is given by [17]

$$\Delta t' = \frac{4\pi R^2 \Omega}{c^2(1 - \frac{R^2 \Omega^2}{c^2})}. \quad (24)$$

which, in the Galilean limit $\beta = \frac{\Omega R}{c} \ll 1$, expanded in terms of β gives

$$\Delta t' = 4\pi R^2 \frac{\Omega}{c^2} (1 + \beta^2 + \mathcal{O}(\beta^4)) \quad (25)$$

in which R is the radius of the circular path traversed by the two beams. This time difference leads to a phase shift $\delta\phi = \frac{2\pi c \Delta t}{\lambda}$ [17]. The same effect could also be analyzed from a rotating observer's point of view, by using the fact that in such a frame the spacetime metric, although flat, is in a stationary form given by the Galilean transformed metric (8) which has a non-Euclidean spatial sector. In the context of the 1+3 (or threading) formulation of spacetime decomposition, it could be shown that this non-Euclidean character is rooted in the cross term of the corresponding stationary metric [11].

To study this rotational effect in the context of the formalism based on MFT, we use the time transformation relation in (14) between the time intervals of two events corresponding to the departure and arrival of the light beam to the same point on the rotating disk,

$$t_2 - t_1 = \cosh \beta (t'_2 - t'_1) + \frac{R}{c} \sinh \beta (\varphi'_2 - \varphi'_1) \quad (26)$$

where in the inertial frame

$$\varphi'_2 - \varphi'_1 = \Omega(t'_2 - t'_1) = \Omega \Delta t' \quad (27)$$

so by substitution from 24 we have

$$\Delta t = \Delta t' (\cosh \beta + \frac{R\Omega}{c} \sinh \beta) \quad (28)$$

which, in the Galilean limit $\beta = \frac{\Omega R}{c} \ll 1$, expanded in terms of β leads to

$$\Delta t = \frac{4\pi R^2 \Omega}{c^2} (1 + 3\beta^2 + \mathcal{O}(\beta^4)) \quad (29)$$

For the calculation of the same effect in the formalism based on Fermi metric first we use the relation between time coordinates in the inertial frame and the rotating one, namely equation (10) from which the time interval between the two events $(t_1, y_1 = 0)$ and $(t_2, y_2 = 0)$ in the rotating frame is related to the inertial observer's time interval as follows

$$\Delta t = \gamma^{-1} \Delta t' \quad (30)$$

which on substitution of the inertial time interval from (25) and expansion in terms of β we end up with

$$\Delta t = 4\pi R^2 \frac{\Omega}{c^2} (1 + \frac{5}{2}\beta^2 + \mathcal{O}(\beta^4)) \quad (31)$$

Comparison of (29) and (31) shows that the two formalisms agree in the classical value but in the next order they differ by a factor of $1/2$.

C. Length Measurement

The length measurements by accelerated observers and their limitations in the formalism based on the hypothesis of locality (and Fermi coordinates) are discussed in [12] and it is shown that the arclength, subtended by angle Φ between two uniformly rotating points on a circle of radius R (such that $\beta^2 = \frac{R^2 \Omega^2}{c^2} \ll 1$), as measured in the local frame of an observer (frame S) in one of those points is given by

$$l = \frac{1}{\sqrt{1 - \beta^2}} [1 - \frac{3}{4}\beta^2 (1 + \frac{\sin 2\Phi - 8 \sin \Phi}{6\Phi})] l' \quad (32)$$

in which $l' = R\Phi$ is the same arclength as measured by an inertial/laboratory observer (frame S') who measures the contracted length as compared to the same arclength $l'' = \gamma l'$ measured by a comoving inertial observer (frame S''). Expansion to the second order in β gives

$$l = [1 - \frac{1}{4}\beta^2 (1 + (\frac{\sin 2\Phi - 8 \sin \Phi}{2\Phi}))] l' + \mathcal{O}(\beta^4) \quad (33)$$

which for small Φ (such that $\sin \Phi \approx \Phi$) reduces to,

$$l = (1 + \frac{1}{2}\beta^2) l' + \mathcal{O}(\beta^4) \quad (34)$$

corresponding to length dilation. But when the circumference is found by setting $\Phi = 2\pi$ the observer finds the following relation

$$L = (1 - \frac{1}{4}\beta^2)2\pi R + \mathcal{O}(\beta^4) \quad (35)$$

In other words small arclengths are dilated in the rotating frame but the whole path is contracted and this is so because for comoving inertial observers the circular path is momentarily an ellipse whose semi-minor axis is along the direction of the observer's motion (see Fig. 4 in [12]). But it should be noted that the same observer, attached to a rotating disk, finds his distance from the center of the disk to be always equal to the ellipse's semi-major axis and hence on returning to the initial point (on the underlying spacetime) he finds out that he is moving on a circular path with radius equal to the semi-major axis of the instantaneous ellipse.

Obviously in the above scenario one could take the two points separated by the arclength, to be on the rim of a uniformly rotating disk of radius R . With this set up one can use the formalism based on MFT to find the same relation between the arclengths as measured by an inertial observer and a rotating one in the Galilean limit [23]. This can be obtained directly from the inverse angular transformation in (14) by setting $\Delta t = 0$ as follows

$$l' \equiv R d\phi' = \cosh(\beta) R d\phi \equiv \cosh(\beta) l \quad (36)$$

in which one can think of l as the small arclength subtended by the observer's open feet, with both ends measured simultaneously. Expanding the above relation to the second order in β we end up with,

$$l = (1 + \frac{1}{2}\beta^2)l' + \mathcal{O}(\beta^4) \quad (37)$$

which compared with (34) shows that to the second order in β , the two formalisms agree on the relation between the small arclengths as measured by the rotating (non-inertial) and inertial observer, and the difference shows up at the fourth order.

To calculate the circumference of the disk, the rotating observer finds the following relation between his/her measurement and that of the inertial observer

$$L = (1 + \frac{1}{2}\beta^2)2\pi R + \mathcal{O}(\beta^4) \quad (38)$$

in other words employing MFT in the rotating frame, the circumference of the path (disk) is dilated which is just the opposite to the relation obtained in (35).

VI. DISCUSSION

In the present paper first we introduced two fundamenatly different approaches to study rotational phenomena. The first approach employs consecutive Lorentz transformation on the basis of the so called hypothesis of locality in which the accelerated frames are taken to be instantaneously equivalent to inertial frames which have the velocity of the accelerated frame at each moment on its worldline. This is in essence special relativity (and local Lorentz transformation) applied to accelerated frames. In the second approach coordinate transformation between inertial and rotating frames is given by a *relativistic rotational transformation* which is intrinsicly different from Lorentz transformation which applies to inertial frames at uniform relative motion. Two well know roational effects, transverse Doppler effect and Sagnac effect, were studied at the Galilean limit ($\beta \ll 1$) in both formalisms to find out what would be their predictions. In both effects the difference in the predicted values as compared to the Galilean values, starts at the second order in β , and that is why the Galilean rotational transformation is a good approximation and applicable at small, everyday rotational velocities. But obviously it is at the relativistic velocities $\beta \rightarrow 1$ that one expects to see deviations from the Galilean predictions. At these volocities the predictions in the two formalisms agree up to the second order in β for transverse Doppler effect whereas for Sagnac effect they differ by a factor of $1/2$. In the measurements of small arclengths the two formalisms agree up to the second order in β and they differ at the fourth order and higher. The two methods disagree on the relation between the circumference of the circular path as measured by an inertial observer and a rotating one at non-zero radius. Measurements of the rotating observer compared to the inertial one, imply contraction of the circumference in the formalism based on Fermi metric and dilation in the formalism based on MFT.

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- [21] For general restrictions on possible extended reference frames for accelerated observers refer

to [6].

- [22] To prevent any confusion it should be noted that up to now Ω was representing the spinning angular velocity of a frame but hereafter it will represent its orbital angular velocity. It should also be noted that in the setup given in figure 1 (e.g for a frame fixed on the rim of a rotating disk) the two angular velocities have the same value.
- [23] Note that in both formalisms we have avoided discussing the Ehrenfest's paradox by only working with the coordinate transformations and not their corresponding spatial metrics (geometries), hence $l = \Phi R$ is taken to be the arclength measured by the inertial (laboratory) observer. For discussions on Ehrenfest's paradox in the same context refer to [13]